B-III Analysis IV Midterm examination 21-02-2017.

Answer all the 8 questions. Each question is worth 5 points.

If you are using any result proved in the class, you need to state it correctly.

1. Let (X, d) and (Y, ρ) be metric spaces. On the product space $X \times Y$, consider the function $r((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + \rho(y_1, y_2)^2}$. Show that r is a metric.

2. Let (X, d) be a metric space and let $A \subset X$ be a compact set with more than 2 points. Show that there exists $a, b \in A$ such that the diameter of A, $dia(A) = d(a, b)$.

3. Show that the set of irrational numbers with the usual metric is a separable metric space.

4. Let (X, d) be a complete metric space. Let $A \subset X$ be a totally bounded set. Show that $\overline{A}$ is a compact set.

5. Let (X, d) be a compact metric space. Let $\{U_\alpha\}_{\alpha \in \Delta}$ be a family of open sets such that $X = \cup_{\alpha \in \Delta} U_\alpha$. Give detailed proof to show that there is a countable set $A \subset \Delta$ such that $X = \cup_{\alpha \in A} U_\alpha$.

6. Let $\mathcal{F} = \{f \in C([0, 1]) : sup_{t \in [0,1]} |f(t)| \leq 1\}$. Show that $\mathcal{F}$ is not an equicontinuous set.

7. Give an example with all the details, of metric spaces (X, d) and (Y, ρ) and a uniformly continuous map $f : X \to Y$ for which there **does not** exist a $0 < c < 1$ so that $\rho(f(x_1), f(x_2)) \leq cd(x_1, x_2)$ for all $x_1, x_2 \in X$.

8. Let $M = \{f \in C([0, 1]) : f(0) = 0\}$. Let $\epsilon > 0$. (*) Show that there is a polynomial p with $p(0) = 0$ and $sup_{t \in [0,1]} |f(t) - p(t)| \leq \epsilon$. (*) $f \in M$.